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INTEGRAL FORM OF THE GENERAL SOLUTION OF EQUATIONS OF STEADY-STATE THERMOELASTICITY*

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A new integral formula is obtained for solving the equation of steady-state thermoelasticity in a three-dimensional region, differing from the well-known formula /1/ in containing no volume integral. A similar formula is encountered in the case of a two-dimensional region, and its use in constructing the integral equation for boundary value problems is suggested. The fact that there are no volume integrals in the integral equations facilitates their numerical solution. If the temperature is represented by Green's formula in terms of the Newtonian potentials of the single and double layer, and the mass force is conservative, then, as shown below, the volume integrals will also be transformed into surface integrals over the boundary surface. The resulting formula however is less suitable for the numerical solution of boundary value problems as it contains a large number of integrals with different kernels.

1. The differential equations of equilibrium of a thermoelastic medium written in terms of the displacements u_i $(i = 1, 2, 3)$ have the form

$$\mu\Delta u_i + (\mu+\lambda)\frac{\partial^2 u_j}{\partial x_i \partial x_j} = \frac{\alpha E}{1-2\nu}\frac{\partial T}{\partial x_i} - K_i \tag{1.1}$$

Here μ and λ are Lamé constants, E and ν is Young's modulus and Poisson's ratio, α is the coefficient of linear thermal expansion and K_i is the mass force density vector. The temperature T is sought in the form of the solution of an independent boundary value problem for the Laplace equation, and is assumed known. We write the solution of (1.1) in the form /1/

$$u_j(x) = \int_S [p_i(y)\, u_{ij}(x, y) - u_i(y)\, p_{ij}(x, y)]\, dS_y + \tag{1.2}$$

$$\int_D K_i(y)\, u_{ij}(x, y)\, dy + \frac{\alpha E}{1-2\nu}\int_D T(y)\,\theta_j(x, y)\, dy$$

$$u_{ij}(x, y) = \frac{(3-4\nu)\delta_{ij} + \beta_i\beta_j}{16\pi\mu(1-\nu) r}, \quad \theta_j(x, y) = \frac{\partial u_{ij}}{\partial y_j}$$

$$p_{ij}(x, y) = \frac{1-2\nu}{8\pi(1-\nu) r^2}\left(n_i\beta_j - n_j\beta_i - \delta_{ij}\cos\varphi - \frac{3\beta_i\beta_j\cos\varphi}{1-2\nu}\right)$$

Here $x(x_1, x_2, x_3)$ and $y(y_1, y_2, y_3)$ denote arbitrary points of the closed region $\bar{D}$, β_i are the direction cosines of the vector $r_i = y_i - x_i$ (r is its modulus), n_i are the direction cosines of the outward normal to the boundary S, φ is the angle between the vector with components r_i and the normal, $p_i(y)$ are the stress vector components on the surface with normal $\{n_i\}$, $dy = dy_1 dy_2 dy_3$ is the volume element of the region D, δ_{ij} is the Kronecker delta. We will write the Green identity for the function T and $\partial r/\partial y_j$ as follows:

$$\int_D \left[T\Delta\left(\frac{\partial r}{\partial y_j}\right) - (\Delta T)\frac{\partial r}{\partial y_j}\right] dy = \frac{\partial}{\partial x_j}\int_S \left(\frac{\partial T}{\partial n} r - T\frac{dr}{\partial n}\right) dS \tag{1.3}$$

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The first term in the integrand on the left hand side of the identity (1.3) is proportional to the function $\theta_j(x, y) = k\Delta(\partial r/\partial y_j)$, $k = (1-2\nu)/[16\pi\mu(1-\nu)]$, and the second term is equal to zero since the temperature T is harmonic. We can therefore rewrite (1.3) in the form

$$\frac{\partial\Phi}{\partial x_j} = \frac{\alpha E}{1-2\nu}\int_D T(y)\,\theta_j(x, y)\,dy \tag{1.4}$$

$$\left(\Phi = \frac{\alpha E}{16\pi\mu(1-\nu)}\int_S \left(\frac{\partial T}{\partial n} r - T\frac{dr}{dn}\right) dS\right)$$

where Φ is a biharmonic function (the sum of the potentials of the single and double layer). The right-hand side of (1.4) is a volume integral which appears in the integral representation (1.2). We note that the derivative (1.4) of the function Φ is a particular solution of the equations of thermoelasticity in the form /2/ $u_j^T = \partial\Phi/\partial x_j$. Using this representation in (1.1), we obtain

$$\Delta\Phi = \alpha\frac{1+\nu}{1-\nu}T \tag{1.5}$$

Differentiating the expression for the function Φ in the integrand we find that (1.5) becomes an identity and the temperature T can be represented by Green's formula

$$T = \frac{1}{4\pi}\int_S \left(\frac{dT}{dn}\frac{1}{r} - T\frac{d}{dn}\left(\frac{1}{r}\right)\right) dS \tag{1.6}$$

Now we can replace the integral representation (1.2) by

$$u_j(x) = \int_S [p_i(y)\,u_{ij}(x, y) - u_i(y)\,p_{ij}(x, y)]\,dS + \int_D K_i(y)\,u_{ij}(x, y)\,dy + \tag{1.7}$$

$$\frac{\alpha E}{16\pi\mu(1-\nu)}\int_S \left(T\frac{\partial^2 r}{\partial y_j\,\partial n} - \frac{dT}{dn}\frac{\partial r}{\partial y_j}\right) dS$$

where the volume integral of T is replaced by a boundary surface integral.

The integral formula (1.7) contains a volume integral of mass force. Let us assume that the mass force K_i is conservative.

$$K_i(y) = -\partial\Psi/\partial y_i$$

and the Laplacian of the force function $\Delta\Psi = m = \mathrm{const}$ (this is the case for the gravitational forces in a homogeneous medium). Then, using Gauss's theorem we obtain

$$\int_D K_i(y)\,u_{ij}(x, y)\,dy = \frac{1-2\nu}{16\pi\mu(1-\nu)} \times \int_S \left(\Psi\frac{\partial^2 r}{\partial y_i\,\partial n} - \frac{d\Psi}{dn}\frac{\partial r}{\partial y_i} + mrn_j\right) dS_y - \int_S \Psi(y)\,n_i(y)\,u_{ij}(x, y)\,dS_y \tag{1.8}$$

Substituting the value of (1.8) into the integral formula (1.7), we obtain complete agreement with the formula given in /3/ without derivation.

2. We assume that the Goodier potential Φ is obtained as a function of the coordinates x_i, i.e. it is not necessarily represented in the form of the sum of biharmonic potentials of a single and double layer. We write the solution of (1.1) in the form

$$u_i = u_i^\circ + u_i^T, \quad u_i = \partial\Phi/\partial x_i \tag{2.1}$$

where u_i° denotes the general part of the solution containing the arbitrary functions and u_i^T is a particular Goodier solution. The elements σ_{ij} of the stress tensor and components p_i of the force vector have a form analogous to (2.1)

$$\sigma_{ij} = \sigma_{ij}^\circ + \sigma_{ij}^T, \quad p_i = p_i^\circ + p_i^T \tag{2.2}$$

The functions σ_{ij}^T, p_i^T appearing in (2.2) are obtained by differentiating the Goodier potential (using formulas known from the theory of elasticity /1/). We introduce the functions u_i^T, σ_{ij}^T, p_i^T as the elements of the basic state, and the Betti-Maizel identity, and use the Kelvin solution as the elements of the auxilliary state. Then in place of (1.4) we obtain

$$\frac{\alpha E}{1-2\nu}\int_D T(y)\,\theta_j(x, y)\,dy = u_j^T(x) - \int_S [p_i^T(y)\,u_{ij}(x, y) - u_i^D(y)\,p_{ij}(x, y)]\,dS_y \tag{2.3}$$

Substituting the representation (2.3) into the integral formula (1.2) we obtain a new integral representation for the solution of thermoelastic equations not containing volume integrals of temperature. We note that (2.3) contains the elastic potentials of the single and double layer which can be combined with the analogous terms of (1.2).

The volume integral of mass forces K_i in (1.2) can also be represented as a sum of integrals over the boundary S of region D, in a form differing from (1.8). To do this we shall seek a particular solution of (1.1) when $\partial T/\partial x_i \equiv 0$, by analogy with the Goodier solution, in the form

$$u_i^K = \partial U/\partial x_i \tag{2.4}$$

where U is a scalar function. The mass forces will be assumed, as before, to be conservative

$$K_i = -\partial \Psi/\partial x_i \tag{2.5}$$

Substituting (2.4) and (2.5) into (1.1), we obtain

$$(\lambda + 2\mu)\frac{\partial}{\partial x_i}(\Delta U) = \frac{\partial \Psi}{\partial x_i} \tag{2.6}$$

When condition

$$\Delta U = \frac{\Psi}{\lambda + 2\mu} \tag{2.7}$$

holds, (2.6) becomes an identity. Let us assume that a particular solution of (2.7) has been found. Then we can find the displacements u_i^K and forces p_i^K

$$p_i^K = 2\mu\left(\frac{\partial^2 U}{\partial y_i\, \partial n} - n_i \Delta U\right) \tag{2.8}$$

Substituting (2.8) into the integral formula (1.2) we obtain, at $T \equiv 0$, a new expression

$$\int_D K_i(y)\, u_{ij}(x, y)\, dy = u_j^K(x) + \int_S [u_i^K(y)\, p_{ij}^K(x, y) - p_i(y)\, u_{ij}(x, y)]\, dS_y \tag{2.9}$$

Formula (2.9) contains, like (2.3), the elastic potentials of the single and double layer, and this enables us to obtain a new integral formula of the theory of thermoelasticity

$$u_j(x) = \int_S [p_i^*(y)\, u_{ij}(x, y) - u_i^*(y)\, p_{ij}(x, y)]\, dS_y \tag{2.10}$$

$$u_i^* = u_i - u_i^T - u_i^K, \quad p_i^* = p_i - p_i^T - p_i^K$$

analogous to the Somigliana formula.

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